Connection probabilities for Ising model and their relation to Dyson's circular ensemble

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Outline

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Ising model

Ising model [Lenz 1920]

A model for ferromagnet, to understand the phase transition.

- $G = (V, E)$ a finite graph
- $\sigma \in \{\ominus, \oplus\}^V$
- $H(\sigma) = -\sum_{x \sim y} \sigma_x \sigma_y$

Ising model is the probability measure of inverse temperature $\beta > 0$:

$$\mu_{\beta,G}[\sigma] \propto \exp(-\beta H(\sigma))$$

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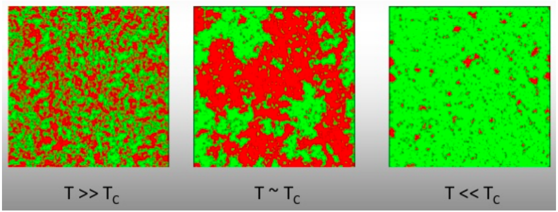
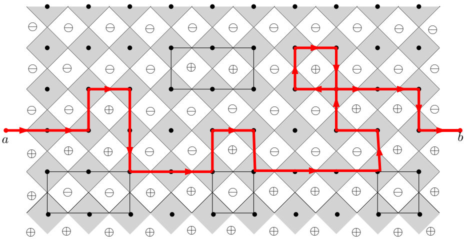


- $\beta > \beta_c$: ordered
- $\beta \approx \beta_c$: critical
- $\beta < \beta_c$: chaotic

Conformal invariance of interfaces



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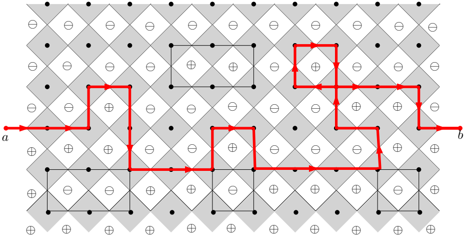

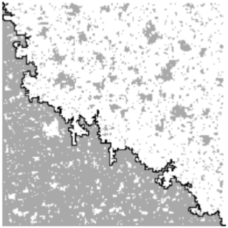

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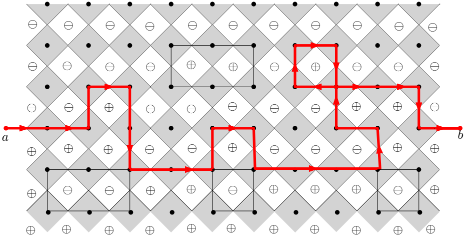
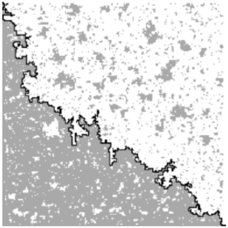


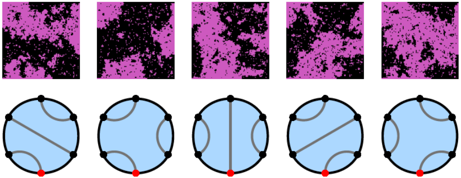


Stanislav Smirnov

Theorem [Chelkak-Smirnov et al. Invent. 2012]

The interface in critical Ising model on $\mathbb{Z}^2$ with Dobrushin boundary conditions converges weakly to SLE_3.

Connection probabilities for Ising model



Theorem [Peltola-W. AAP 2023]

The connection of Ising interfaces forms a planar link pattern $\mathcal{A}_\delta$.

$$\lim_{\delta\to 0} \mathbb{P}[\mathcal{A}_\delta = \alpha] = \frac{\mathcal{Z}_\alpha(\Omega; x_1, \ldots, x_{2N})}{\mathcal{Z}_{Ising}(\Omega; x_1, \ldots, x_{2N})}, \quad \mathcal{Z}_{Ising} = \sum_{\alpha\in \mathrm{LP}_N} \mathcal{Z}_\alpha,$$

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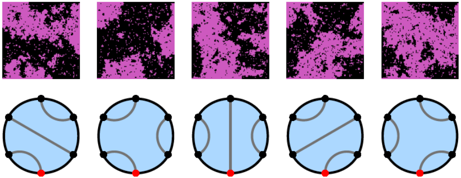

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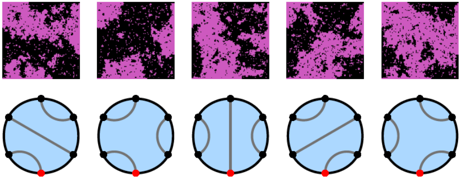

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- Related to correlation functions in CFT.

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PDE : $\left[\frac{\kappa}{2}\partial_i^2 + \sum_{j\neq i}\left(\frac{2}{x_j-x_i}\partial_j - \frac{2h}{(x_j-x_i)^2}\right)\right]\mathcal{Z}(x_1,\ldots,x_{2N}) = 0$, where $h=(6-\kappa)/2\kappa$.

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- PDE : 2N variables, 2N PDEs
- ASY : boundary value ?

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Questions

Existence and uniqueness ?

Pure partition functions

Uniqueness [Flores-Kleban, CMP 2015]

Fix $\kappa \in (0, 8)$. If there exist collections of smooth functions satisfying PDE, COV and ASY, they are (essentially) unique.

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Existence

- $\kappa \in (0, 8) \setminus \mathbb{Q}$ [Kytölä-Peltola, CMP 2016]
- $\kappa \in (0, 4]$ [Peltola-W. CMP 2019, Beffara-Peltola-W. AOP 2021]
- $\kappa \in (0, 6]$ [W. CMP 2020]

- Coulumb gas techniques
- Global multiple SLEs
- Hypergeometric SLE

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Theorem [W. CMP 2020]

Fix $\kappa \in (0,6]$. The pure partition functions are the recursive collection $\{\mathcal{Z}_\alpha : \alpha \in \cup_N \mathrm{LP}_N\}$ of smooth functions $\mathcal{Z}_\alpha : \mathfrak{X}_{2N} \to \mathbb{R}$ uniquely determined by the following properties :

PDE, COV, ASY as well as **PLB** :

$$0 < \mathcal{Z}_\alpha(x_1, \ldots, x_{2N}) \le \prod_{\{a,b\} \in \alpha} |x_b - x_a|^{-2h}, \quad \forall (x_1, \ldots, x_{2N}) \in \mathfrak{X}_{2N}.$$

$\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}_N\}$ is linearly independent and forms a basis for the solution space.

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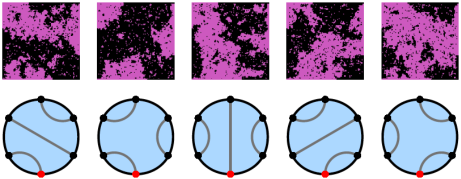

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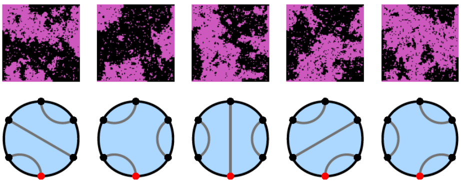

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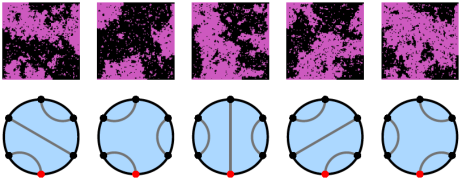

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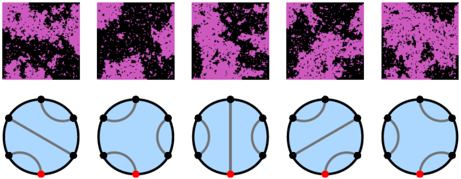

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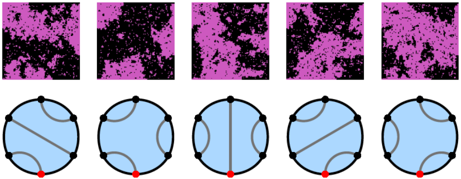

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Consequence : given the connectivity α, a single interface $\sim$ Loewner chain associated to $\mathcal{Z}_\alpha$:

$$\mathrm{d}W_t=\mathrm{d}B_t+\partial_j\log\mathcal{Z}_\alpha(V_t^1,\ldots,V_t^{j-1},W_t,V_t^{j+1},\ldots,V_t^{2N})\mathrm{d}t.$$

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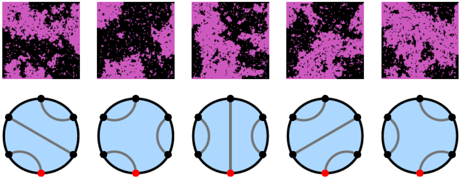
Usual parameterization vs common parameterization



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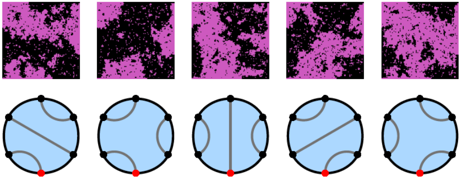

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From the upper-half plane to the unit disc :

$$\mathcal{G}_\alpha(\theta^1, \ldots, \theta^{2N}) = \mathcal{Z}_\alpha(\mathbb{D}; \exp(2\mathrm{i}\theta^1), \ldots, \exp(2\mathrm{i}\theta^{2N})),$$

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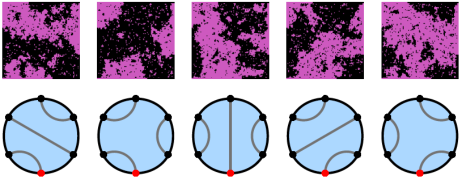

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Fix $a = 2/\kappa$. Under a-common parameterization :

$$\mathrm{d}\theta_t^j = \mathrm{d}B_t^j + \partial_j \log \mathcal{G}_\alpha(\theta_t^1, \ldots, \theta_t^{2N})\mathrm{d}t + a\sum_{k\neq j} \cot(\theta_t^j - \theta_t^k)\mathrm{d}t, \quad 1 \leq j \leq 2N, t < T,$$

where $\{B^j\}_{1\leq j\leq 2N}$ are independent Brownian motions and T is the collision time.

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Proposition [Feng-W.-Yang 2023]

Fix $\kappa \in (0, 4]$ and $a = 2/\kappa$. The solution $\boldsymbol{\theta}_t = (\theta_t^1, \ldots, \theta_t^{2N})$ to (1) conditioned on $\{T > s\}$ converges in total variation distance as $s \to \infty$ to $2N$ radial Bessel process

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whose invariant density is called Dyson's circular ensemble [Dyson, J. Math. Phys. 1962] :

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Key ingredients : [Peltola-W. CMP 2019], [Healey-Lawler, PTRF 2021], $2N$-time local martingale :

$$M_{\boldsymbol{t}}^{\alpha} = g_{\boldsymbol{t}}'(0)^{-2N\tilde{b}} \prod_{j=1}^{2N} h_{\boldsymbol{t},j}'(\xi_{t_j}^j)^b g_{\boldsymbol{t},j}'(0)^{\tilde{b}} \times \mathcal{G}_\alpha(\theta_{\boldsymbol{t}}^1, \ldots, \theta_{\boldsymbol{t}}^{2N}) \exp\left(\frac{c}{2} \sum_{j=1}^{2N} \mu_{\boldsymbol{t}}^j\right).$$

Applications : multiple Ising interfaces in annulus

We consider critical Ising model in annulus with boundary conditions :

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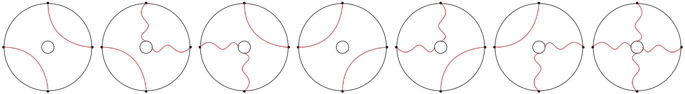

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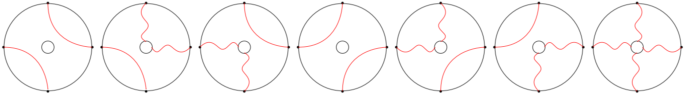

Theorem [Feng-W.-Yang 2023]

Suppose $(\eta_1, \ldots, \eta_{2N}) \sim \mathbb{P}^{(\boldsymbol{\theta})}_{\text{Ising}}$. We have

$$\mathbb{P}^{(\boldsymbol{\theta})}_{\text{Ising}}[\eta_1, \ldots, \eta_{2N} \text{ all hit } r\mathbb{D}] = r^{\frac{16N^2-1}{24}+o(1)}, \quad \text{as } r \to 0. \tag{4}$$

Conditioned on the event $\{\eta_1, \ldots, \eta_{2N} \text{ all hit } r\mathbb{D}\}$, the law $\mathbb{P}^{(\boldsymbol{\theta})}_{\text{Ising}}$ converges in total variation distance to $2N$-sided radial SLE_3 whose driving function is $2N$ radial Bessel process

$$\mathrm{d}\theta^j_t = \mathrm{d}B^j_t + \frac{4}{3}\sum_{k\neq j}\cot(\theta^j_t - \theta^k_t)\mathrm{d}t, \quad 1 \le j \le 2N. \tag{5}$$

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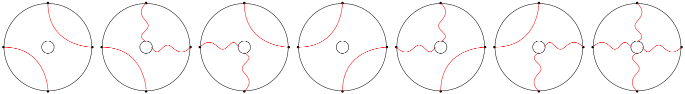

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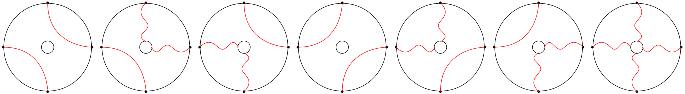

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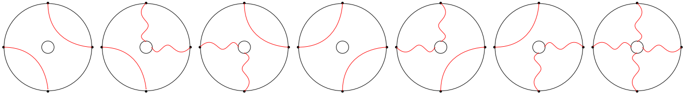

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- Eq. (5) : [Cardy, J. Phys. A 2003]

Thanks !

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